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p-Adic Green Functions and Zeta Functions

by

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§0. Introduction

Let $V = \mathbb{Q}_p^n$ be the n-dimensional vector space over the p-adic number field $\mathbb{Q}_p$ and V^* its dual vector space. For a polynomial function P^* on V^* with coefficients in $\mathbb{Q}_p$, we can define a p-adic pseudodifferential operator $P^*(\partial)_p$ by

$$\phi \longmapsto P^*(\partial)_p\phi = \mathcal{F}^{-1}(|P^*|_p\mathcal{F}(\phi)),$$

where $\mathcal{F} : L^2(V) \to L^2(V^*)$ is the Fourier transform and $|*|_p$ denotes the p-adic norm. The function $|P^*|_p$ is called the *symbol* of $P^*(\partial)_p$. By abuse of language, we also call P^* the symbol. The operator $P^*(\partial)_p$ has a self-adjoint extension with dense domain in $L^2(V)$ and is considered to be a p-adic analogue of linear partial differential operator with constant coefficients.

With a motivation coming from the so-called p-adic quantum field theory, Vladimirov and Volovich [VV] raised the problem of studying the Green function

$$G = \mathcal{F}^{-1}\left(\frac{1}{|P^*|_p + m^2}\right), \quad m > 0$$

for the symbol $P^* = y_1^2 + \cdots + y_n^2$. In this case $P^*(\partial)_p$ can be viewed as a p-adic analogue of $-\Delta$ ($\Delta =$ the n-dimensional Laplacian) and the function G plays the role of a fundamental solution of the equation

$$(P^*(\partial)_p + m^2)\phi = f.$$

As is seen in a recent work of Jang [J], the Green function G is closely related to the zeta function

$$Z^*(\phi, s) = \int_{\{y\in V^* \mid P^*(y)\neq 0\}} |P^*(y)|_p^s\phi(y)\,dy, \quad \phi \in \mathcal{S}(V^*).$$

In particular the functional equation satisfied by $Z^*(\phi, s)$ plays a decisive role in [J].

In this paper we consider the Green function G in a more general setting where the zeta function attached to the symbol P^* is deeply studied, namely in the case where the symbol P^* is a relative invariant of some reductive regular prehomogeneous vector space (for the theory of prehomogeneous vector spaces, see [Sm], [SS], [SK], [I2], [Sf], [G]).

After some discussion in §1 on elementary properties of the operator $P^*(\partial)_p$ for general P^*, we restrict our attention to the operators obtained from relative invariants of reductive regular prehomogeneous vector spaces and prove the following (Theorem 2.3):

(1) The Green function G is a locally constant function on an open dense subset Ω of V; more precisely, there exists a polynomial $P(x)$ on V (the dual of P^*) such that Ω is the complement of the zero locus of $P(x)$ and $G(x)$ is a function of $|P(x)|_p$ on Ω.

(2) G has a (rather complicated) convergent series expansion valid on Ω.

(3) If $|P(x)|_p$ is sufficiently large, then G has a convergent power series expansion of the form

$$G(x) = \frac{1}{m^2|P_0(x)|_p^{1/2}} \sum_{\alpha=1}^{\infty} \gamma(\alpha)(-m^2|P(x)|_p)^{-\alpha}$$

for some constants $\gamma(\alpha)$.

What is remarkable here is that the Green function is completely controlled by the gamma factor of the functional equation connecting the zeta function attached to the symbol P^* to the orbital zeta functions attached to P. In fact the constants $\gamma(\alpha)$ are the values of the gamma factor at positive integer arguments. Note that the whole argument can easily be extended to the case where the base field is an arbitrary finite algebraic extension of $\mathbb{Q}_p$.

If the symbol is given by $P^* = y_1^2 + \cdots + y_n^2$, then the results (1), (2) above were obtained by Vladimirov and Volovich [VV] for $n = 1$, by Bikulov [B] for $n = 2$ and $p \geq 3$, and by Jang [J] for even $n \geq 3$, while the result (3) is new also in this special case, since these earlier results are concerned only with the first term of the power series expansion in (3). Some related results, which corresponds to the case $m = 0$, are discussed in [Ko1] and [Ko2, §2].

The classification theory of prehomogeneous vector spaces (see e.g., [SK], [K], [KKY], [KKMO], [KKMM]) gives a lot of examples of symbols P^* to which the results above can apply. Here are some samples:

(i) $V = \mathbb{Q}_p^n$, $P^* =$ a quadratic form,

(ii) $V = M_n(\mathbb{Q}_p)$, $P^* = \det x$,

(iii) $V = Sym_n(\mathbb{Q}_p) = \left\{x \in M_n(\mathbb{Q}_p) \;\middle|\; {}^t x = x\right\}$, $P^* = \det x$,

(iv) $V = Alt_{2n}(\mathbb{Q}_p) = \left\{x \in M_{2n}(\mathbb{Q}_p) \;\middle|\; {}^t x = -x\right\}$, $P^* =$ the Pfaffian of x,

(v) $V = M_{m,n}(\mathbb{Q}_p)$ $(m > n)$, $P^* = \det {}^t x x$.

The cases (i) and (ii) will be examined in §3.

A problem left to be solved is the investigation of the behaviour of the Green function near $V \setminus \Omega$, the zero locus of $P(x)$. The Green function has in general a singularity on $V \setminus \Omega$ of the form $|P(x)|_p^{-\lambda}(\log|P(x)|_p)^m$ and the analysis of this singularity is much harder than the analysis of the behaviour at infinity done in the present paper. It is a conjecture that the values of the gamma factor at negative integer arguments control the asymptotic behaviour of the Green function $G(x)$ when x approaches to $V \setminus \Omega$. In §4, we examine the conjecture for the simplest case, namely, the case of one variable. The conjecture has turned out to be true by a recent work of Ochiai [O], which will appear soon.

As a closing comment of this introduction, we point out that this kind of application of prehomogeneous vector spaces is not surprising at all, since the original motivation of M. Sato, the initiater of the theory, was the explicit construction of fundamental solutions of linear partial differential equations.

A large part of the present paper is based on a talk of the author in a seminar conducted by Professors Endo and Khrennikov in 1996. The author would like to thank them.

§1. A p-adic analogue of linear partial differential operators and the Green functions

We denote by $\mathbb{Q}_p$ the p-adic number field and by $\mathbb{Z}_p$ the ring of p-adic intergers. Let $|\ |_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ the absolute value on $\mathbb{Q}_p$ normalized so that $|p|_p = p^{-1}$. Let $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be the additive character defined by

$$\begin{array}{ccccccc} \psi : \mathbb{Q}_p & \to & \mathbb{Q}_p/\mathbb{Z}_p & \hookrightarrow & \mathbb{Q}/\mathbb{Z} & \to & \mathbb{C}\,. \\ & & & & a & \mapsto & \exp(2\pi i a) \end{array}$$

We normalize the Haar measure $d\alpha$ on $\mathbb{Q}_p$ such that $\int_{\mathbb{Z}_p} d\alpha = 1$.

Let $V = \mathbb{Q}_p^n$ be the n-dimensional $\mathbb{Q}_p$-vector space and V^* its dual vector space. We identify V^* with $\mathbb{Q}_p^n$ via the standard inner product

$$(x, y) = x_1 y_1 + \cdots + x_n y_n \quad (x \in V = \mathbb{Q}_p^n,\ y \in V^* = \mathbb{Q}_p^n)\,.$$

The Haar measure dx on $\mathbb{Q}_p^n$ is given by $dx = dx_1 \cdots dx_n$ (dx_i = the normalized Haar measure on $\mathbb{Q}_p$), which is autodual with respect to the pairing $\psi((x, y))$.

We denote the space of all Schwartz-Bruhat functions on $V = \mathbb{Q}_p^n$ by $\mathcal{S}(V)$. For $\phi \in \mathcal{S}(V)$, we define its Fourier transform $\mathcal{F}\phi$ by

$$\mathcal{F}\phi(y) = \int_V \phi(x)\psi(-(x, y))\, dx\,.$$

Then the Fourier transform induces a linear isomorphism of $\mathcal{S}(V)$ onto $\mathcal{S}(V^*)$ and the inverse transform is given by

$$\mathcal{F}^{-1}\phi^*(x) = \int_{V^*} \phi^*(y)\psi((x, y))\, dy \quad (\phi^* \in \mathcal{S}(V^*))\,.$$

The Fourier transform $\mathcal{F}$ can be extended to an isometry of $L^2(V)$ onto $L^2(V^*)$.

Now, for a polynomial function $P^* = P^*(y_1, \cdots, y_n)$ on V^* with coefficients in $\mathbb{Q}_p$, we can define an operator $P^*(\partial)_p$ that acts on functions in

$$Dom = \{\phi \in L^2(V) \mid |P^*(y)|_p \cdot \mathcal{F}\phi \in L^2(V^*)\}$$

by

$$(1.1) \qquad (P^*(\partial)_p\phi)(x) = \mathcal{F}^{-1}(|P^*(y)|_p\mathcal{F}\phi)(x) = \int_{V^*} |P^*(y_1, \cdots, y_n)|_p\mathcal{F}\phi(y)\psi((x, y))\, dy\,.$$

If ϕ is in $\mathcal{S}(V)$, then the integrand of the right hand side of (1.1) is a continuous function with compact support and $\mathcal{S}(V)$ is contained in Dom. Moreover the image $P^*(\partial)_p\psi$ is then uniformly locally constant and $P^*(\partial)_p$ induces a mapping

$$P^*(\partial)_p : \mathcal{S}(V) \to C^\infty_{\mathrm{unif}}(V)\,,$$

where $C^\infty_{\mathrm{unif}}(V)$ is the space of all functions on V invariant under translation of every element in $(p^N\mathbb{Z}_p)^n$ for some sufficiently large N.

For functions ϕ_1, ϕ_2 on V, we define a $\mathbb{C}$-bilinear form $\langle\phi_1, \phi_2\rangle$ and a hermitian inner product (ϕ_1, ϕ_2) by setting

$$\langle\phi_1, \phi_2\rangle = \int_V \phi_1(x)\phi_2(x)\, dx\,, \quad (\phi_1, \phi_2) = \int_V \phi_1(x)\overline{\phi_2(x)}\, dx\,.$$

LEMMA 1.1. (1) *For any $\phi_1, \phi_2 \in Dom$, we have*

$$\langle P^*(\partial)_p\phi_1, \phi_2\rangle = \langle\phi_1, P^*(\partial)_p\phi_2\rangle \quad \textit{and} \quad (P^*(\partial)_p\phi_1, \phi_2) = (\phi_1, P^*(\partial)_p\phi_2)\,.$$

(2) *The operator $P^*(\partial)_p$ is a symmetric positive operator, hence has a self-adjoint extension.*

Proof. The lemma is an immediate consequence of the identity

$$(P^*(\partial)_p\phi_1, \phi_2) = (|P^*|_p\mathcal{F}\phi_1, \mathcal{F}\phi_2)\,. \qquad \square$$

Let $\mathcal{S}'(V)$ be the space of distributions on V, namely, the space of linear forms on $\mathcal{S}(V)$. As usual, any locally integrable function f on V can be identified with the distribution defined by

$$\mathcal{S}(V) \ni \phi \longmapsto \int_V f(x)\phi(x)\, dx \in \mathbb{C}.$$

Denote by $\mathcal{S}'(V)_P$ the space of linear forms on $P^*(\partial)_p\mathcal{S}(V) + \mathcal{S}(V) \subset C^\infty_{\mathrm{unif}}(V)$. Then, in view of Lemma 1.1 (1), we define the distribution $P^*(\partial)_pT$ for $T \in \mathcal{S}'(V)_P$ by setting

$$\langle P^*(\partial)_pT, \phi\rangle = \langle T, P^*(\partial)_p\phi\rangle \quad (\phi \in \mathcal{S}(V))\,.$$

Now we are interested in the Green function of the following pseudodifferential equation

$$(1.2) \qquad (P^*(\partial)_p + m^2)\Phi = \Psi, \quad m > 0\,.$$

DEFINITION. If a distribution $G \in \mathcal{S}'(V)_P$ satisfies the equation

$$(1.3) \qquad (P^*(\partial)_p + m^2)G = \delta\,,$$

then G is called a *Green function*. Here δ is the Dirac delta function.

PROPOSITION 1.2. *The distribution*

$$G = \mathcal{F}^{-1}\left(\frac{1}{|P^*|_p + m^2}\right)$$

is a Green function.

Proof. By the definition of the action of $P^*(\partial)_p$ on distributions, we have

$$\begin{aligned}\langle (P^*(\partial)_p + m^2)G, \phi\rangle &= \langle G, (P^*(\partial)_p + m^2)\phi\rangle \\ &= \left\langle \frac{1}{|P^*|_p + m^2}, \mathcal{F}(P^*(\partial)_p + m^2)\phi \right\rangle \\ &= \left\langle \frac{1}{|P^*|_p + m^2}, (|P^*|_p + m^2)\mathcal{F}\phi \right\rangle \\ &= \langle 1, \mathcal{F}\phi\rangle = \phi(0)\,.\end{aligned}$$

This proves the proposition. □

Now let us see that the Green function G constructed in the proposition above plays the role of a fundamental solution of the equation (1.2).

PROPOSITION 1.3. *For a $\Psi \in \mathcal{S}(V)$ and $x \in V$, we define a function $\Psi_x \in \mathcal{S}(V)$ by $\Psi_x(z) = \Psi(x - z)$ ($z \in V$). If we put $\Phi(x) = \langle G, \Psi_x\rangle$, then the function Φ is a solution of the equation*

$$(P^*(\partial)_p + m^2)\Phi = \Psi\,.$$

Proof. In the following we denote by ch_A the characteristic function of a subset A of V. It is enough to prove the proposition for $\Psi = ch_{a+B_N}$ ($a \in V,\ N > 0$), where we put $B_N = (p^N\mathbb{Z}_p)^n$. Let us calculate $\Phi = \langle G, \Psi_x\rangle$. Since $\Psi_x = ch_{x-a+B_N}$, we have

$$\begin{aligned}\langle G, \Psi_x\rangle &= \left\langle \frac{1}{|P^*|_p + m^2}, \mathcal{F}\Psi_x \right\rangle \\ &= \left\langle \frac{1}{|P^*(y)|_p + m^2}, p^{-nN}\psi((a - x, y))ch_{B_{-N}} \right\rangle \\ &= p^{-nN}\int_{B_{-N}} \frac{\psi((a - x, y))}{|P^*(y)|_p + m^2}\,dy\,.\end{aligned}$$

Hence Φ is the inverse Fourier transform of the function

$$p^{-nN}\frac{\psi((a, y))}{|P^*(y)|_p + m^2}ch_{B_{-N}}(y)\,.$$

Therefore we obtain

$$\begin{aligned}(P^*(\partial)_p + m^2)\Phi(x) &= \mathcal{F}^{-1}\left((|P^*|_p + m^2)\cdot p^{-nN}\frac{\psi((a, *))}{|P^*|_p + m^2}ch_{B_{-N}}\right) \\ &= \mathcal{F}^{-1}(p^{-nN}\psi((a, *))ch_{B_{-N}}) \\ &= ch_{a+B_N} = \Psi.\end{aligned}$$

This prove the proposition. □

REMARK. Let G' be another Green function. Then the distribution $T = G - G'$ satisfies the equation $(P^*(\partial)_p + m^2)T = 0$, hence $(|P^*|_p + m^2)\mathcal{F}T = 0$. Since $|P^*|_p + m^2$ is a nowhere vanishing locally constant function on $\Omega^* = \{ y \in V^* \mid P^*(y) \neq 0\}$, we have $\mathcal{F}T|_{\Omega^*} = 0$. Therefore we

$$\mathcal{F}G'|_{\Omega^*} = \mathcal{F}G|_{\Omega^*} = \frac{1}{|P^*|_p + m^2} .$$

Thus the Green function G given in Proposition 1.2 is the most natural one.

For the investigation of the Green function G, it is useful to consider the function $\mathcal{G}$ defined by the improper integral

$$\mathcal{G}(x) = \lim_{N\to\infty} G_N(x), \quad G_N(x) := \int_{(p^{-N}\mathbb{Z}_p)^n} \frac{\psi(-(x, y))}{|P^*(y)|_p + m^2}\, dy . \tag{1.4}$$

The following lemma gives a relation between the distribution G and the function $\mathcal{G}$.

LEMMA 1.4. *Let Ω be the open subset of V consisting of all points x satisfying the condition*

there exist an neighbourhood U of x and a positive integer N such that

$$G_N(u) = G_{N+1}(u) = G_{N+2}(u) = \cdots , \quad (\forall u \in U) .$$

Then the distribution G restricted to Ω coincides with the locally constant function $\mathcal{G}$ defined by the improper integral (1.4). *Namely, we have*

$$\langle G, \phi\rangle = \int_V \mathcal{G}(x)\phi(x)\, dx \quad (\phi \in \mathcal{S}(\Omega)) .$$

Proof. It is obvious that the improper integral defines a locally constant function $\mathcal{G}$ on Ω. For any open compact subset K of Ω, one can choose a positive integer N such that $\mathcal{G}(x) = G_N(x)$ for all $x \in K$. Hence, for any $\phi \in \mathcal{S}(\Omega)$, there exists an N such that

$$\int_V \mathcal{G}(x)\phi(x)\, dx = \int_V G_N(x)\phi(x)\, dx . \tag{1.5}$$

Therefore we have

$$\begin{aligned}
\langle G, \phi\rangle &= \int_{V^*} \frac{\mathcal{F}\phi(y)}{|P^*(y)|_p + m^2}\, dy = \lim_{N\to\infty} \int_{(p^{-N}\mathbb{Z}_p)^n} \frac{\mathcal{F}\phi(y)}{|P^*(y)|_p + m^2}\, dy \\
&= \lim_{N\to\infty} \int_V \phi(x)\, dx \int_{(p^{-N}\mathbb{Z}_p)^n} \frac{\psi(-(x, y))}{|P^*(y)|_p + m^2}\, dy \\
&= \lim_{N\to\infty} \int_V G_N(x)\phi(x)\, dx = \lim_{N\to\infty} \int_V \mathcal{G}(x)\phi(x)\, dx \\
&= \int_V \mathcal{G}(x)\phi(x)\, dx .
\end{aligned}$$

This proves the lemma. □

§2. Green functions for relative invariants

In this section, we examine the Green function

$$G(x) = \mathcal{F}^{-1}\left(\frac{1}{|P^*(y)|_p + m^2}\right) \tag{2.1}$$

in the case where the symbol P^* is a relative invariant of a reductive regular prehomogeneous vector space.

First we recall some basic definitions in the theory of prehomogeneous vector spaces. Let $\mathbf{G}$ be a connected linear algebraic group defined over $\mathbb{Q}_p$ and ρ a rational representation of $\mathbf{G}$ defined over $\mathbb{Q}_p$ on the affine n-space $\mathbf{V}$. The triple $(\mathbf{G}, \rho, \mathbf{V})$ is called a *prehomogeneous vector space* if there exists a proper algebraic subset $\mathbf{S}$ such that $\mathbf{V}(\overline{\mathbb{Q}}_p) - \mathbf{S}(\overline{\mathbb{Q}}_p)$ is a single $\mathbf{G}(\overline{\mathbb{Q}}_p)$-orbit, where $\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. The algebraic set $\mathbf{S}$ is called the *singular set* of $(\mathbf{G}, \rho, \mathbf{V})$.

We fix a coordinate system on $\mathbf{V}$. Then we may consider the representation ρ as a matrix representation $\rho : \mathbf{G} \to GL(n)$. We define the dual representation ρ^* by $\rho^*(g) = {}^t\rho(g)^{-1}$ $(g \in \mathbf{G})$. We denote by $\mathbf{V}^*$ the affine n-space viewed as the representation space of ρ^*.

In the following we assume that $(\mathbf{G}, \rho, \mathbf{V})$ is reductive and regular. This means that

$$\mathbf{G} \text{ is reductive and } \mathbf{S} \text{ is a hypersurface}. \tag{2.2}$$

Then the dual triple $(\mathbf{G}, \rho^*, \mathbf{V}^*)$ is also a prehomogeneous vector space and the singular set $\mathbf{S}^*$ is a hypersurface.

Since the singular set $\mathbf{S}$ is automatically defined over $\mathbb{Q}_p$, we can find a polynomial P with coefficients in $\mathbb{Q}_p$ such that $\mathbf{S} = \{x \in \mathbf{V} \mid P(x) = 0\}$. Then P is known to be a relative invariant of $\mathbf{G}$. Namely there exists a rational character $\chi : \mathbf{G} \to GL(1)$ defined over $\mathbb{Q}_p$ such that

$$P(\rho(g)x) = \chi(g)P(x) \quad (x \in \mathbf{V},\ g \in \mathbf{G}).$$

Moreover there exists another polynomial P^* (unique up to constant factor) satisfying

$$P^*(\rho^*(g)y) = \chi(g)^{-1}P^*(y) \quad (y \in \mathbf{V}^*,\ g \in \mathbf{G})$$

and then $\mathbf{S}^* = \{y \in \mathbf{V}^* \mid P^*(y) = 0\}$ is the singular set of $(\mathbf{G}, \rho^*, \mathbf{V}^*)$.

We put $V = \mathbf{V}(\mathbb{Q}_p)$ and $V^* = \mathbf{V}^*(\mathbb{Q}_p)$ and take the polynomial P^* obtained in this manner as the symbol of our pseudodifferential operator $P^*(\partial)_p$.

Now let us introduce the zeta functions attached to the polynomials P and P^*, which play a crucial role in our study of the Green function.

Put

$$\Omega = \{x \in V \mid P(x) \neq 0\}, \quad \Omega^* = \{y \in V^* \mid P^*(y) \neq 0\}.$$

Let

$$\Omega = V_1 \cup \cdots \cup V_\nu, \quad \Omega^* = V_1^* \cup \cdots \cup V_\nu^*$$

be the $\mathbf{G}(\mathbb{Q}_p)$-orbit decomposition. It is known that Ω and Ω^* decompose into the same finite number of $\mathbf{G}(\mathbb{Q}_p)$-orbits.

The orbital zeta functions attached to the polynomials P and P^* are defined as follows:

$$Z_i(\phi; s) = \int_{V_i} |P(x)|_p^s |P_0(y)|_p^{-1/2} \phi(x)\, dx\,, \quad (\phi \in \mathcal{S}(V))\,,$$

$$Z_i^*(\phi^*; s) = \int_{V_i^*} |P^*(y)|_p^s \phi^*(y)\, dy\,, \quad (\phi^* \in \mathcal{S}(V^*))$$

$(i = 1, \cdots, \nu)$, where s is a complex parameter and P_0 is the relative invariant of $(\mathbf{G}, \rho, \mathbf{V})$ satisfying

$$P_0(\rho(g)x) = \det \rho(g)^2 P_0(x) \quad (x \in \mathbf{V},\ g \in \mathbf{G})\,.$$

By the assumption (2.2), such a P_0 exists and is unique up to a non-zero constant factor.

The following is the fundamental theorem in the theory of prehomogeneous vector spaces over the p-adic number field.

THEOREM 2.1 (1) *The integrals $Z_i(\phi; s)$ and $Z_i^*(\phi^*; s)$ $(i = 1, \cdots, \nu)$ are absolutely convergent for sufficiently large* Re(s) *and have analytic continuations to meromorhic functions of s in $\mathbb{C}$. More precisely, $Z_i(\phi; s)$ and $Z_i^*(\phi^*; s)$ $(i = 1, \cdots, \nu)$ represent rational functions of p^{-s}.*

(2) *For any $\phi \in \mathcal{S}(V)$, the following functional equations hold*

$$Z_j^*(\mathcal{F}\phi; s) = \sum_{i=1}^{\nu} \gamma_{ji}(s) Z_i(\phi; -s) \quad (j = 1, \cdots, \nu)\,,$$

where $\gamma_{ij}(s)$ are rational functions of p^{-s} independent of ϕ.

The theorem is proved by Gyoja in a still unpublished work [G]. A proof under the so-called finite orbit condition (but for not necessarily reductive prehomogeneous vector spaces) is found in [Sf]. If $\mathbf{S}$ is absolutely irreducible and $\mathbf{G}$ is self-adjoint, then the theorem is due to Igusa [I2].

For the application to the Green function, it is necessary to consider the zeta function

$$Z^*(\phi^*; s) = \sum_{j=1}^{\nu} Z_j^*(\phi^*; s)\,.$$

Then $Z^*(\phi^*; s)$ satisfies the functional equation

$$Z^*(\mathcal{F}\phi; s) = \sum_{i=1}^{\nu} \gamma_i(s) Z_i(\phi; -s), \quad \gamma_i(s) = \sum_{j=1}^{\nu} \gamma_{ji}(s) \tag{2.3}$$

for any $\phi \in \mathcal{S}(V)$.

It will turn out that the functions $\gamma_i(s)$ control the Green function and the following is the key lemma in the proof of our main theorem.

LEMMA 2.2. *The functions $\gamma_i(s)$ $(i = 1, \cdots, \nu)$ are of the form*

$$\frac{\sum_{j=-\ell_1}^{\ell_2} c_j p^{js}}{\prod_{k=1}^{r}(1-p^{-a_k(s+\lambda_k)})}, \tag{2.4}$$

where a_k are positive integers and λ_k are positive rational numbers. In particular we have

$$\gamma_i(s) = O(p^{\alpha|\mathrm{Re}(s)|}), \quad \alpha = \begin{cases} \ell_2 & \text{as } \mathrm{Re}(s) \to +\infty, \\ \ell_1 - \sum_{k=1}^{r} a_k & \text{as } \mathrm{Re}(s) \to -\infty. \end{cases}$$

Proof. Fix a point a in V_i and consider an open neighbourhood $a + (p^N \mathbb{Z}_p)^n$ in V_i. Assume that N is so large that $|P|_p$ and $|P_0|_p$ are constant on the neighbourhood. Let ϕ_0 be the characteristic function of $a + (p^N \mathbb{Z}_p)^n$ and put $\phi^* = \mathcal{F}\phi_0$. Then, by (2.3), we have

$$\gamma_i(s) = \frac{Z^*(\phi^*; s)}{Z_i(\phi_0; -s)} = p^{-nN} |P_0(a)|_p^{1/2} |P(a)|_p^s Z^*(\phi^*; s).$$

Hence the lemma (except the positivity of λ_k) follows immediately from the corresponding result for $Z^*(\phi^*; s)$ (see Lemma 2.1 in [Sf]). Since the integral $Z^*(\phi^*; s)$ is absolutely convergent for $\mathrm{Re}(s) \geq 0$, the zeta function has no poles in an open set containing $\mathrm{Re}(s) \geq 0$. This implies that λ_k are positive, if the expression (2.4) is reduced. From (2.4), the estimate of $\gamma_i(s)$ is obvious. □

THEOREM 2.3. (1) *The Green function G is a locally constant function on $\Omega = \{x \in V \mid P(x) \neq 0\}$. Moreover, for a point $x \in \Omega$, let V_i be the $\mathbf{G}(\mathbb{Q}_p)$-orbit containing x. Then we have the following expression as absolutely convergent series*:

$$G(x) = |P_0(x)|_p^{-1/2} \sum_{\mu=-\ell_2}^{\infty} \frac{A_\mu |P(x)|_p}{m^2 |P(x)|_p + p^{-\mu}}, \tag{2.5}$$

where A_μ are the coefficients of the expansion

$$\gamma_i(s) = \sum_{\mu=-\ell_2}^{\infty} A_\mu p^{-\mu s} \quad (\mathrm{Re}(s) > 0). \tag{2.6}$$

(2) *If $|P(x)| > p^{\ell_2}/m^2$, then we have the following expression as absolutely convergent series*:

$$G(x) = \frac{1}{m^2 |P_0(x)|_p^{1/2}} \sum_{\alpha=1}^{\infty} \gamma_i(\alpha)(-m^2 |P(x)|_p)^{-\alpha}. \tag{2.7}$$

REMARK. (1) The second part of the theorem gives an asymptotic expansion of the Green function as $|P(x)|_p \to \infty$ in V_i. From the identity (2.7), one can see that the coefficients of the asymptotic expansion depend on the $\mathbf{G}(\mathbb{Q}_p)$-orbit V_i. This can be viewed as an analogue of the Stokes phenomenon (see Example 2 in §3).

(2) Note that the expansion (2.7) coincides with the identity derived from the following formal argument:

$$
\begin{aligned}
G|_{V_i} &= \mathcal{F}^{-1}\left(\frac{1}{|P^*|_p + m^2}\right)\bigg|_{V_i} = \frac{1}{m^2}\mathcal{F}^{-1}\left(\frac{1}{1+|P^*|_p/m^2}\right)\bigg|_{V_i} \\
&= \frac{1}{m^2}\mathcal{F}^{-1}\left(\sum_{\alpha=0}^{\infty}(-|P^*|_p/m^2)^\alpha\right)\bigg|_{V_i} = \frac{1}{m^2}\sum_{\alpha=0}^{\infty}\mathcal{F}^{-1}(-|P^*|_p/m^2)^\alpha|_{V_i} \\
&= \frac{1}{m^2}\sum_{\alpha=0}^{\infty}(-m^2)^{-\alpha}\mathcal{F}^{-1}(|P^*|_p^\alpha)|_{V_i} \\
&= \frac{1}{m^2|P_0|_p^{1/2}}\sum_{\alpha=0}^{\infty}(-m^2)^{-\alpha}\gamma_i(\alpha)|P|_p^{-\alpha}\,.
\end{aligned}
$$

As we shall see later in the proof of Theorem 2.3, we have $\gamma_i(0) = 0$. Hence the last expression coincides with (2.7).

Before starting the proof of Theorem 2.3, let us prove the following estimation of A_μ, from which the convergence of the series (2.5) follows immediately.

LEMMA 2.4. *$A_\mu = O(p^{-\beta\mu})$ for any $\beta < \min\{\lambda_1, \cdots \lambda_r\}$.*

Proof. We define B_μ by

$$
\frac{1}{\prod_{k=1}^r(1-p^{-a_k(s+\lambda_k)})} = \sum_{\mu=0}^{\infty} B_\mu p^{-\mu s} \quad (\mathrm{Re}(s) > 0)\,.
$$

Then, since

$$
A_\mu = \sum_{j=-\ell_1}^{\ell_2} c_j B_{\mu+j}\,,
$$

the lemma follows from the estimate $B_\mu = O(p^{-\beta\mu})$. By the definition of B_μ, we have

$$
B_\mu = \sum_{\substack{\mu_1,\cdots,\mu_r\ge 0\\ a_1\mu_1+\cdots+a_r\mu_r=\mu}} p^{-(a_1\mu_1\lambda_1+\cdots+a_r\mu_r\lambda_r)}\,.
$$

As is easily seen, it is sufficient to estimate B_μ under the assumption $a_1 = \cdots = a_r = 1$. We prove the estimate by induction on r. Since $\lambda_k > 0$, the lemma is obvious for $r = 1$. Assume that $r \ge 2$ and $\beta < \min\{\lambda_1, \cdots \lambda_r\}$. Then, by the induction hypothesis, we have

$$
\begin{aligned}
B_\mu &= \sum_{\mu_r=0}^{\mu} p^{-\mu_r\lambda_r} \sum_{\mu_1+\cdots+\mu_{r-1}=\mu-\mu_r} p^{-(\mu_1\lambda_1+\cdots+\mu_{r-1}\lambda_{r-1})} \\
&< C_1 \sum_{\mu_r=0}^{\mu} p^{-\mu_r\lambda_r}\cdot p^{-\beta(\mu-\mu_r)}
\end{aligned}
$$

$$< C_1 p^{-\beta\mu} \sum_{\mu_r=0}^{\mu} p^{-\mu_r(\lambda_r-\beta)} .$$

for some C_1. Hence we have

$$B_\mu < C_1 p^{-\beta\mu} \frac{1-p^{-(\mu+1)(\lambda_r-\beta)}}{1-p^{-(\lambda_r-\beta)}} < \frac{C_1}{1-p^{-(\lambda_r-\beta)}} \cdot p^{-\beta\mu} .$$

This proves the lemma. □

Proof of Theorem 2.3. (1) First we consider the integral

$$G_N(x) = \int_{(p^{-N}\mathbb{Z}_p)^n} \frac{\psi(-(x,y))}{|P^*(y)|_p + m^2}\, dy$$

for $x \in V_i$. By Fubini's theorem, we have

$$\begin{aligned} G_N(x) &= \int_{(p^{-N}\mathbb{Z}_p)^n} \psi(-(x,y))\, dy \int_0^\infty e^{-(|P^*(y)|_p+m^2)\theta}\, d\theta \\ &= \int_0^\infty e^{-m^2\theta}\, d\theta \int_{(p^{-N}\mathbb{Z}_p)^n} \psi(-(x,y)) e^{-\theta|P^*(y)|_p}\, dy . \end{aligned}$$

Using the Taylor expansion of e^x, one can easily see that the last expression is equal to

$$\int_0^\infty e^{-m^2\theta}\, d\theta \int_{(p^{-N}\mathbb{Z}_p)^n} \psi(-(x,y)) \sum_{\alpha=0}^\infty \frac{(-\theta)^\alpha}{\alpha!} |P^*(y)|_p^\alpha\, dy .$$

Again, by Fubini's theorem, we can exchange the summation with respect to α and the integration with respect to y. Hence we obtain

$$G_N(x) = \int_0^\infty e^{-m^2\theta} \left(\sum_{\alpha=0}^\infty \frac{(-\theta)^\alpha}{\alpha!} \int_{(p^{-N}\mathbb{Z}_p)^n} \psi(-(x,y))|P^*(y)|_p^\alpha\, dy \right) d\theta .$$

Denote by $ch_{-N}(y)$ the characteristic function of $(p^{-N}\mathbb{Z}_p)^n$ and put

$$\phi^*_{N,y}(y) = \psi(-(x,y)) ch_{-N}(y) .$$

Then, since

$$Z^*(\phi^*_{N,x}, \alpha) = \int_{(p^{-N}\mathbb{Z}_p)^n} \psi(-(x,y))|P^*(y)|_p^\alpha\, dy .$$

it follows from Theorem 2.1 that

$$G_N(x) = |P_0(x)|_p^{-1/2} \int_0^\infty e^{-m^2\theta} \left(\sum_{\alpha=0}^\infty \frac{(-\theta)^\alpha}{\alpha!} \sum_{j=1}^{\nu} \gamma_j(\alpha) Z_j(\mathcal{F}^{-1}\phi^*_{N,x}; -\alpha) \right) d\theta .$$

An elementary calculation shows that

$$\mathcal{F}^{-1}\phi^*_{N,x}(z) = p^{nN} ch_N(-x+z) = p^{nN} \times \text{the characteristic function of } x + (p^N\mathbb{Z}_p)^n .$$

Hence, if N is sufficiently large, then

$$Z_j(\mathcal{F}^{-1}\phi^*_{N,x}; -\alpha) = \begin{cases} |P_0(x)|_p^{-1/2}|P(x)|_p^{-\alpha} & (j = i)\,, \\ 0 & (j \neq i)\,. \end{cases}$$

Note that the right hand side does not depend on N and a locally constant function of x on Ω. Hence, by Lemma 1.4, the distribution G restricted to Ω is a locally constant function and is given by

$$(2.8)\quad G(x) = \lim_{N\to\infty} G_N(x) = |P_0(x)|_p^{-1/2} \int_0^\infty e^{-m^2\theta} \left(\sum_{\alpha=0}^{\infty} \gamma_i(\alpha) \frac{(-\theta/|P(x)|_p)^\alpha}{\alpha!} \right) d\theta\,.$$

By (2.6), we have

$$G(x) = |P_0(x)|_p^{-1/2} \int_0^\infty e^{-m^2\theta} \left(\sum_{\alpha=0}^{\infty} \sum_{\mu=-\ell_2}^{\infty} A_\mu \cdot \frac{(-\theta p^{-\mu}/|P(x)|_p)^\alpha}{\alpha!} \right) d\theta\,.$$

The estimate of A_μ in Lemma 2.4 implies that the double sum with respect to α and μ in the right hand side is absolutely convergent. Hence we can exchange the order of the summations and get

$$\begin{aligned} G(x) &= |P_0(x)|_p^{-1/2} \int_0^\infty e^{-m^2\theta} \sum_{\mu=-\ell_2}^{\infty} A_\mu \exp(-\theta p^{-\mu}/|P(x)|_p)\, d\theta \\ &= |P_0(x)|_p^{-1/2} \sum_{\mu=-\ell_2}^{\infty} A_\mu \int_0^\infty \exp(-\theta(m^2 + p^{-\mu}/|P(x)|_p))\, d\theta \\ &= |P_0(x)|_p^{-1/2} \sum_{\mu=-\ell_2}^{\infty} \frac{A_\mu}{m^2 + p^{-\mu}/|P(x)|_p}\,. \end{aligned}$$

This proves the first part of the theorem.

(2) Now we prove the second part. Notice that $Z^*(\phi^*_{N,x}, 0) = 0$ for sufficiently large N, since $x \mapsto \psi(-(x, y))$ gives a nontrivial character of $(p^{-N}\mathbb{Z}_p)^n$. Hence $\gamma_i(0) = 0$ and, by (2.8), we obtain

$$\begin{aligned} G(x) &= \lim_{T\to\infty} |P_0(x)|_p^{-1/2} G(T, x), \\ G(T, x) &= \int_0^T e^{-m^2\theta} \left(\sum_{\alpha=1}^{\infty} \gamma_i(\alpha) \frac{(-\theta/|P(x)|_p)^\alpha}{\alpha!} \right) d\theta\,. \end{aligned}$$

Since $\gamma_i(\alpha) = O(p^{\ell_2\alpha})$ $(\alpha \to +\infty)$ by Lemma 2.2, the integral defining $G(T, x)$ is absolutely convergent and, by Fubini's theorem, we have

$$G(T, x) = \sum_{\alpha=1}^{\infty} \gamma_i(\alpha) \frac{(-|P(x)|_p)^{-\alpha}}{\alpha!} \int_0^T \theta^\alpha e^{-m^2\theta}\, d\theta\,.$$

We note here that

$$\frac{1}{\alpha!}\int_0^T \theta^\alpha e^{-m^2\theta}\,d\theta < \frac{1}{\alpha!}\int_0^\infty \theta^\alpha e^{-m^2\theta}\,d\theta = \frac{1}{m^{2(\alpha+1)}}\frac{\Gamma(\alpha+1)}{\alpha!} = \frac{1}{m^{2(\alpha+1)}}\,.$$

Therefore, if $|P(x)|_p > p^{\ell_2}/m^2$, then the convergence of $G(T,x)$ is uniform with respect to T and

$$\begin{aligned}G(x) &= \lim_{T\to\infty} |P_0(x)|_p^{-1/2} G(T,x)\\ &= |P_0(x)|_p^{-1/2}\sum_{\alpha=1}^{\infty}\gamma_i(\alpha)\frac{(-|P(x)|_p)^{-\alpha}}{\alpha!}\lim_{T\to\infty}\int_0^T \theta^\alpha e^{-m^2\theta}\,d\theta\\ &= \frac{1}{m^2|P_0(x)|_p^{1/2}}\sum_{\alpha=1}^{\infty}\gamma_i(\alpha)(-m^2|P(x)|_p)^{-\alpha}\,.\end{aligned}$$

Now the theorem is completely proved. □

§3. Examples

In the following, $\mathbf{V}(n)$ denotes the affine n-space.

3.1. The case $P^*(y) = y^2$

The simplest prehomogeneous vector space is

$$(\mathbf{G},\rho,\mathbf{V}) = (GL(1),\rho,\mathbf{V}(1))\,,\quad \rho(g)x = gx \quad (g\in GL(1),\ x\in\mathbf{V}(1))\,.$$

Then $\mathbf{S} = \{0\}$ and $P_0(x) = x^2$. We put $P(x) = x^2$. The dual representation ρ^* is given by

$$\rho^*(g)y = g^{-1}y \quad (g\in GL(1),\ y\in\mathbf{V}(1))$$

and

$$\mathbf{S}^* = \{0\},\quad P^*(y) = y^2\,.$$

The zeta functions attached to P and P^* are given by

$$Z(\phi,s) = \int_{\mathbb{Q}_p^\times}|x|_p^{2s-1}\phi(x)\,dx\,,\quad Z^*(\phi^*,s) = \int_{\mathbb{Q}_p^\times}|y|_p^{2s}\phi^*(y)\,dy$$

and the functional equation reads

$$Z^*(\mathcal{F}\phi,s) = \gamma(s)Z(\phi,-s)\,,\quad \gamma(s) = \frac{1-p^{2s}}{1-p^{-2s-1}} = -p^{2s} + (1-p^{-1})\sum_{\mu=0}^{\infty}p^{-\mu}p^{-2\mu s}$$

(see, e.g., [T]). Hence, by Theorem 2.3 (1), we have

$$G(x) = |x|_p\left(\frac{-1}{m^2|x|_p^2+p^2} + (1-p^{-1})\sum_{\mu=0}^{\infty}\frac{p^{-\mu}}{m^2|x|_p^2+p^{-2\mu}}\right)\quad (x\neq 0)\,.$$

It is easy to see that the right hand side is equal to

$$(1-p^{-1})\frac{|x|_p}{m^2|x|_p^2+p^2}\sum_{\mu=0}^{\infty} p^{-\mu}\frac{p^2-p^{-2\mu}}{m^2|x|_p^2+p^{-2\mu}}\,,$$

which is the expression given in [VV].

If $|x|_p > p/m$, by Theorem 2.3 (2), we have

$$G(x)=\frac{1}{m}\sum_{\alpha=1}^{\infty}(-1)^{\alpha}\frac{1-p^{2\alpha}}{1-p^{-2\alpha-1}}(m|x|_p)^{-(2\alpha+1)}\,.$$

The term corresponding to $\alpha = 1$ was obtained also in [VV].

3.2. The case of quadratic forms

Let Y be a nondegenerate symmetric matrix of size n with entries in $\mathbb{Q}_p$ and consider the prehomogeneous vector space

$$(\mathbf{G},\rho,\mathbf{V})=(GL(1)\times SO(Y),\rho,\mathbf{V}(n))\,,$$
$$\rho(t,h)x=thx \quad (t\in GL(1),\ h\in SO(Y),\ x\in\mathbf{V}(n))\,.$$

Then the singular set is given by $\mathbf{S}=\{x\in\mathbb{Q}_p^n \mid P(x)=0\}$ with $P(x)={}^txYx$ and $P_0(x)=P(x)^n$. The dual representation ρ^* is given by

$$\rho^*(t,h)y=t^{-1\,t}h^{-1}y \quad (g\in GL(1),\ h\in SO(Y)\ y\in\mathbf{V}(n))$$

and

$$\mathbf{S}^*=\{y\in\mathbb{Q}_p^n \mid P^*(y)=0\}\,,\qquad P^*(y)={}^tyY^{-1}y\,.$$

We define the zeta functions attached to P^* by

$$Z^*(\phi^*,s)=\int_{\Omega^*}|P^*(y)|_p^s\phi^*(y)\,dy\,.$$

In the present case we have to introduce orbital zeta functions attached to P. The $GL(1)_{\mathbb{Q}_p}\times SO(Y)_{\mathbb{Q}_p}$-orbit decomposition of Ω is given by

$$\Omega=\bigcup_{\varepsilon\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2}V_\varepsilon,\qquad V_\varepsilon=\{x\in\mathbb{Q}_p^n \mid P(x)\in\varepsilon\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2\}.$$

We put

$$Z_\varepsilon(\phi,s)=\int_{V_\varepsilon}|P(x)|_p^{s-n/2}\phi(x)\,dx\,.$$

The functional equation satisfied by these zeta functions was calculated by Rallis and Schiffmann [RS, Théorème 2-13]. To describe it in a form convenient to our purpose, we need some notational preliminaries. Put $D=\det Y$, $D^*=(-1)^{[n/2]}D$, and

$$\rho(s)=\frac{1-p^{s-1}}{1-p^{-s}}\,.$$

For $\eta \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$, we define $\rho_\eta(s)$ by the following table:

	$p > 2$	$p = 2$
$\mathrm{ord}(\eta) \equiv 0 \bmod 2$	$\dfrac{1-p^{-1}}{2} \cdot \dfrac{1}{1-p^{-2s}}$	$\dfrac{1}{8}\left\{\dfrac{1}{1-2^{-2s}} + 2^{2s}\exp\left(\dfrac{\eta_0 \pi i}{2}\right)\right\}$
$\mathrm{ord}(\eta) \equiv 1 \bmod 2$	$-\dfrac{p^{s-1}}{2} \cdot \dfrac{1-p^{-2s+1}}{1-p^{-2s}} + \dfrac{p^{(2s-1)/2}\sigma_p}{2}\left(\dfrac{\eta_0}{p}\right)$	$\dfrac{1}{8}\left\{-2^s \cdot \dfrac{1-2^{-2s+1}}{1-2^{-2s}} + 2^{3s}\exp\left(\dfrac{\eta_0\pi i}{4}\right)\right\}$

Here we put $\eta_0 = \eta/p^{\mathrm{ord}(\eta)}$, (η_0/p) is the Legendre symbol and $\sigma_p = 1$ or $\sqrt{-1}$ according as $p \equiv 1 \bmod 4$ or $3 \bmod 4$. Note that $\rho(s)$ and $\rho_\eta(s)$ are special cases of the Tate gamma factors (see [T]).

We recall that there exists a constant $\gamma(Y)$ (often called *the Weil constant*) with absolute value 1 satisfying

$$\int_{\mathbb{Q}_p^n} \mathcal{F}\phi(x)\psi(P(x))\,dx = |2|^{-n/2}|D|^{-1/2}\gamma(Y)\int_{\mathbb{Q}_p^n}\phi(y)\psi(-4^{-1}P^*(y))\,dy$$

for any $\phi \in \mathcal{S}(\mathbb{Q}_p^n)$ ([W]). The constant $\gamma(Y)$ depends only on the $GL(n, \mathbb{Q}_p)$-equivalence class of Y and, if Y is equivalent to $\mathrm{diag}(a_1, \cdots, a_n)$, then $\gamma(Y) = \gamma(a_1)\cdots\gamma(a_n)$, where we write $\gamma(a)$ for the Weil constant determined by the matrix (a) of size 1. Hence the calculation of $\gamma(Y)$ reduces to the case where $n = 1$. The Weil constant $\gamma(a)$ depends only on the class of a in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. It is known that, for a unit $u \in \mathbb{Z}_p^\times \cap \mathbb{Z}$, we have

$$\gamma(u) = \begin{cases} 1 & (p > 2), \\ \exp\left(\dfrac{\pm\pi i}{4}\right) & (p = 2,\ u \equiv \pm 1 \bmod 4)\,, \end{cases}$$

$$\gamma(up) = \begin{cases} \left(\dfrac{u}{p}\right)\sigma_p & (p > 2)\,, \\ \exp\left(\dfrac{u\pi i}{4}\right) & (p = 2)\,. \end{cases}$$

Now we can describe the functional equation (2.3) explicitly. The result is the following:

$$Z^*(\mathcal{F}\phi, s) = \sum_{\varepsilon} \Gamma_\varepsilon(s) Z_\varepsilon(\phi, -s)\,,$$

where

$$(3.1)\quad \Gamma_\varepsilon(s) = |2|^{2s+n/2}|D|^{1/2}\gamma(Y)(-\varepsilon, D^*)_p\rho(s+1)$$
$$\times \begin{cases} \displaystyle\sum_{\eta\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2} (\eta, D^*)_p\rho_\eta\left(s+\frac{n}{2}\right) & (n\equiv 0 \bmod 2)\,, \\ \gamma(\varepsilon)(-\varepsilon,-1)_p \displaystyle\sum_{\eta\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2} (\eta, -\varepsilon D^*)_p\gamma(\eta)\rho_\eta\left(s+\frac{n}{2}\right) & (n\equiv 1 \bmod 2)\,. \end{cases}$$

Here we denote by $(*,*)_p$ the Hilbert symbol. As we have noticed in Remark (1) to Theorem 2.3, the forms of the two expressions of the Green function $G(x)$ in Theorem 2.3 (1) and (2) may depend on the orbit to which the point x belongs. The formula above for $\Gamma_\varepsilon(s)$ shows that this is the case in the present example.

We give here the explicit expression of the Green function only in the case where $p > 2$ and $n \equiv 0 \bmod 2$. In this case, we get

$$\Gamma_\varepsilon(s) = |D|^{1/2}\gamma(Y)(-\varepsilon, D^*)_p \cdot \frac{1-p^s}{1-p^{-s-1}} \times \begin{cases} \dfrac{1-(p,D^*)_p p^{s+\frac{n}{2}-1}}{1-(p,D^*)_p p^{-s-n/2}}\,, & (\delta, D^*)_p = 1\,, \\ (p, D^*)_p p^{s+(n-1)/2}\sigma_p\,, & (\delta, D^*)_p = -1\,, \end{cases}$$

where δ is a non-square unit. Thus we obtain an explicit formula for the asymptotic expansion of $G(x)$ given in Theorem 2.3 (2):

$$G(x) = \sum_{\alpha=1}^{\infty}(-1)^\alpha m^{-2(\alpha+1)}\Gamma_\varepsilon(\alpha)|P(x)|_p^{-\alpha-n/2}\,, \quad |P(x)|_p > 2m^{-2}\,,\ \ x\in V_\varepsilon\,.$$

In particular, the coefficient of the first term of the asymptotic expansion is given by

$$-m^{-4}|D|^{1/2}\gamma(Y)(-\varepsilon, D^*)_p \cdot \frac{1-p}{1-p^{-2}} \times \begin{cases} \dfrac{1-(p,D^*)_p p^{\frac{n}{2}}}{1-(p,D^*)_p p^{-1-\frac{n}{2}}}\,, & (\delta, D^*)_p = 1\,, \\ (p, D^*)_p p^{\frac{n}{2}+\frac{1}{2}}\sigma_p\,, & (\delta, D^*)_p = -1\,. \end{cases}$$

If $P(x) = x_1^2 + \cdots + x_n^2$, this is due to Jang [J, Theorem 3.4]. (We note that the factor $(-\varepsilon, D^*)_p$ is missing in [J, Theorem 3.4]. This results from an error in his Proposition 3.2.)

For simplicity, put $\lambda = (p, D^*)$. If $(\delta, D^*)_p = 1$, then we have

$$\begin{aligned}\Gamma_\varepsilon(s) =& |D|^{1/2}\gamma(Y)(-\varepsilon, D^*)_p \\ &\times \Bigg\{(\lambda p^{n/2-1})p^{2s} + (1-p^{-1})(\lambda p^{n/2}-1)p^s \\ &+ \sum_{\mu=0}^{\infty}\left(p^\mu\cdot\frac{(1-p)(1-\lambda p^{n/2}) - (1-\lambda p)(1-p^{n/2})p^{-(n/2+1)(\mu+2)}}{1-\lambda p^{-n/2+1}}\right)\cdot p^{-\mu s}\Bigg\}\end{aligned}$$

and, if $(\delta, D^*)_p = -1$, then we obtain

$$\begin{aligned}\Gamma_\varepsilon(s) =& |D|^{1/2}\gamma(Y)(-p\varepsilon, D^*)_p\sigma_p \\ &\times\left(- p^{(n-1)/2}p^{2s} + (1-p^{-1})\sum_{\mu=-1}^{\infty} p^{-\mu+(n-3)/2}p^{-\mu s}\right).\end{aligned}$$

Let $A_{\mu,\varepsilon}$ be the coefficient of $p^{-\mu s}$ in this expansion. Then, by Theorem 2.3 (1), we have

$$G(x) = \sum_{\mu=-2}^{\infty} \frac{A_{\mu,\varepsilon}|P(x)|_p^{-n/2+1}}{m^2|P(x)|_p + p^{-\mu}}, \quad x \in V_\varepsilon.$$

3.3. The case of the determinant

Our third example is the following:

$$(\mathbf{G}, \rho, \mathbf{V}) = (GL(n), \rho, M(n)), \quad \rho(g)x = gx \quad (g \in GL(n),\ x \in M(n)).$$

Then $\mathbf{S} = \{x \in M(n) \mid \det x = 0\}$ and $P_0(x) = \det x^{2n}$. We put $P(x) = \det x$. The dual representation ρ^* is given by

$$\rho^*(g)y = {}^t g^{-1}y \quad (g \in GL(n),\ y \in M(n))$$

and

$$\mathbf{S}^* = \{y \in M(n) \mid \det y = 0\}, \quad P^*(y) = \det y.$$

The zeta functions attached to P and P^* are given by

$$Z(\phi, s) = \int_{\mathbb{Q}_p^\times} |\det x|_p^{s-n}\phi(x)dx, \quad Z^*(\phi^*, s) = \int_{\mathbb{Q}_p^\times} |\det y|_p^{s}\phi^*(y)\,dy$$

and the functional equation reads

$$\begin{aligned}Z^*(\mathcal{F}\phi, s) &= \gamma(s)Z(\phi, -s),\\ \gamma(s) &= (-1)^n p^{ns+n(n-1)/2}\frac{1-p^{-s}}{1-p^{-s-n}}\\ &= (-1)^n p^{n(n-1)/2}\left(p^{ns} + (1-p^n)\sum_{\mu=1}^{\infty} p^{-n\mu}p^{(n-\mu)s}\right)\end{aligned}$$

(see, e.g., [Sf], Theorem 3.4). Hence, by Theorem 2.4 (1), we have for $\det x \neq 0$

$$G(x) = \frac{(-1)^n p^{n(n-1)/2}}{|\det x|_p^n}\left(\frac{1}{m^2|\det x|_p + p^n} + (1-p^n)\sum_{\mu=1}^{\infty}\frac{p^{-n\mu}}{m^2|\det x|_p^2 + p^{-\mu+n}}\right).$$

If $|\det x|_p > p^n/m^2$, by Theorem 2.3 (2), we have

$$G(x) = \frac{(-1)^n p^{n(n-1)/2}}{m^2|\det x|_p^n}\sum_{\alpha=1}^{\infty}\frac{1-p^{-\alpha}}{1-p^{-\alpha-n}}(-m^2p^{-n}|\det x|_p)^{-\alpha}.$$

§4. A remark on the asymptotic behaviour near the singular set

If a point x moves from one orbit V_i to another orbit V_j across the singular set, the zero locus of $P(x)$, then the form of the expansion of the Green function $G(x)$ given in Theorem 2.3 may change. Hence it is interesting to investigate the asymptotic behaviour of $G(x)$ when x approaches to the singular set.

A formal computation similar to the one in Remark (2) to Theorem 2.3 suggests the asymtotic formula

$$(4.1)\qquad G(x) \sim \frac{|P(x)|_p}{|P_0(x)|_p^{1/2}} \sum_{\alpha=0}^{\infty} \gamma_i(-\alpha-1)(-m^2|P(x)|_p)^{\alpha} \quad \text{as } |P(x)|_p \to 0 \quad \text{in } V_i\,.$$

However this formula is not completely correct and a modification is necessary, because of the divergence of the integral defining the zeta functions in the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \ll 0\}$. In particular $\gamma_i(s)$ may have poles at $s = -\alpha - 1$ for some nonnegative integers α. Then the logarithmic function $(\log|P(x)|_p)^k$ would appear as a correction term.

Let us illustrate the situation with the simplest example $P(x) = x$ and $P^*(y) = y$. In this case, the functional equation (2.3) reads

$$\int_{\mathbb{Q}_p^\times} |y|_p^s \mathcal{F}\phi(y)\,dy = \gamma(s) \int_{\mathbb{Q}_p^\times} |x|_p^{-s-1}\phi(x)dx\,, \quad \gamma(s) = \frac{1-p^s}{1-p^{-s-1}}\,.$$

Then we have the following asymptotic expansion of $G(x)$.

PROPOSITION 4.1. *The Green function $G(x)$ for the symbol $P^*(y) = y$ has the asymptotic expansion*

$$G(x) \sim -\frac{1-p^{-1}}{\log p}\cdot \log|x|_p + c_0 + \sum_{\alpha=1}^{\infty} \gamma(-1-\alpha)(-m^2|x|_p)^{\alpha}\,, \quad |x|_p \to 0\,,$$

where

$$c_0 = -p^{-1} + (1-p^{-1})\left(\sum_{i=0}^{\infty} \frac{1}{1+m^2p^i} - \sum_{i=0}^{\infty} \frac{m^2}{m^2+p^i}\right).$$

Note that $P_0(x) = x^2$. Hence the terms corresponding to $\alpha \geq 1$ are precisely the same as the ones predicited by (4.1). Since $\gamma(s)$ has a pole at $s = -1$, a modification is necessary for the term corresponding to $\alpha = 0$ and $\log|x|_p$ appears as a correction term.

We omit the proof of the proposition above, since the general case of relative invariants of reductive regular prehomogeneous vector spaces is treated in a work of Ochiai [O].

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